Theory of hybrid systems. II. The symmetrized product and redefined Lie bracket of quantum mechanics

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Abstract

The symmetrized product for quantum mechanical observables is defined. It is seen as consisting of the ordinary multiplication and the application of the superoperator that orders the operators of coordinate and momentum. This superoperator is defined in a way that allows obstruction free quantization when the observables are considered from the point of view of the algebra. Then, the operatorial version of the Poisson bracket is defined. It is shown that it has all properties of the Lie bracket and that it can substitute the commutator in the von Neumann equation.
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1 Introduction

The Hermitian operators $\hat{q}$ and $\hat{p}$, acting in the Hilbert space of states, represent coordinate and momentum of quantum mechanical system and correspond to the coordinate q and momentum p of classical mechanical system. For these operators it should hold $[\hat{q}, \hat{p}] = i\hbar\hat{I}$. This is one part of the quantization procedure other parts of which are the following. One has to say which operator should correspond to, let say, $q^n p^m$ of classical mechanics. That is where the problem of symmetrized product (ordering rule) of quantum mechanics firstly enters. Namely, one is not willing to accept $\hat{q}^n \cdot \hat{p}^m$ because this operator, in contrast to the involved ones, is not the Hermitian. (With $\cdot$, which will be sometimes omitted, we represent ordinary product - successive application, of operators.) One has to define new operation, denoted by $\circ$, according to which the product of $\hat{q}^n$ and $\hat{p}^m$ will be the Hermitian. The most convenient way for doing that is by using already existing operations of addition, ordinary multiplication and multiplication by numbers. With this new operation the algebraic structure of quantum mechanics appears in its rudimentary form. In Cohen (1966) and Kerner and Sutcliffe (1970) one can find a short review of different propositions for symmetrized product while in Shewell (1959) there is a critical discussion of many ordering rules introduced in there cited references.

On the other hand, the classical mechanical variables have structure of both the algebra and Lie algebra. The Lie product, or bracket, of classical mechanics is the Poisson bracket:

$$\{f(q,p), g(q,p)\} = \frac{\partial f(q,p)}{\partial q} \cdot \frac{\partial g(q,p)}{\partial p} - \frac{\partial g(q,p)}{\partial q} \cdot \frac{\partial f(q,p)}{\partial p}, \tag{1}$$

where $f(q,p)$ and $g(q,p)$ belong to the set of classical variables. Its importance comes from the fact that it appears in the dynamical equation (as well as in all transformations due to symmetries). The expression:

$$\frac{\partial \rho(q,p,t)}{\partial t} = \{H(q,p), \rho(q,p,t)\}, \tag{2}$$

is the Liouville equation where $\rho(q,p,t)$ is distribution representing state and $H(q,p)$ is the Hamiltonian of classical system.

It is desired to equip the set of quantum mechanical observables not just with the algebraic (symmetrized) product, but with the Lie algebraic product as well, *i.e.*, one has to introduce the quantum analogue of the Poisson bracket. After Dirac (1958), the canonical quantization prescription says that the Poisson bracket of classical variables has to be translated into the $\frac{1}{i\hbar}$ times the commutator of corresponding quantum observables. The most important place where the commutator appears is the von Neumann equation:

$$\frac{\partial \hat{\rho}(t)}{\partial t} = \frac{1}{i\hbar}[H(\hat{q},\hat{p}), \hat{\rho}(t)]. \tag{3}$$

It prescribes the way in which the state of quantum system $\hat{\rho}(t)$ evolves due to the Hamiltonian $H(\hat{q},\hat{p})$. (Similar expressions hold for all symmetry transformations.) The last equation, of course, for a pure state ($\hat{\rho}(t)^2 = \hat{\rho}(t)$) reduces to the Schrödinger equation.

Discussions on the algebraic and Lie algebraic structures in quantum mechanics one can find in Emch (1972) and Joseph (1970). In Arens and Babit (1965), Gotay (1980) and (1996) and Chernoff (1995), it was found that these two are interrelated in such a way that there is an obstruction to quantization which is manifested through the existence of some contradiction. Concretely, if the Lie bracket of quantum observables $f_1(\hat{q}, \hat{p})$ and $f_2(\hat{q}, \hat{p})$ is not equal to the Lie bracket of $f_3(\hat{q}, \hat{p})$ and $f_4(\hat{q}, \hat{p})$, while for the Poisson brackets of classical variables, to which these quantum observables should correspond, holds $\{f_1(q,p), f_2(q,p)\} = \{f_3(q,p), f_4(q,p)\}$, then there are two different operators that can be attributed to the same function and quantization can not be taken as selfconsistent. One can conclude, see Chernoff (1995), that the problem of quantization, which we have sketched above, is impossible.

This article is the second in series devoted to the development of theory of hybrid systems. Within it we shall define the symmetrized product of quantum mechanics at the first place. Then, by using this product, we shall be able to propose new Lie bracket of quantum mechanics. It will be the operatorial version of the Poisson bracket. In this way the algebraic and Lie algebraic aspect of quantum and classical mechanics will appear to be the same, as will become obvious latter. This result is of great importance for the foundation of theory of hybrid systems. However, since the ordering rule and Lie bracket are *per se* interesting and deserve detailed consideration, we shall approach them here without reference to the rest of theory of hybrid systems.

2 Definition of the symmetrized product

The most often used example of symmetrized product of two operators is $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$. This operator is the Hermitian and it corresponds to qp of classical mechanics. But, this expression is susceptible of more than one meaning. Namely, one can look on $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$ as on "one half of the anticommutator of $\hat{q}$ and $\hat{p}$" or one may take it as "the sum of all combinations of involved operators divided by the number of this combinations". Different understandings of this generally accepted expression lead to constructions of nonequivalent expressions in other, not so trivial cases. In Kerner and

Sutcliffe (1970) it was shown that there given propositions for symmetrized product start to differ for quartic monomials in $\hat{q}$ and $\hat{p}$.

We believe that by examining the way in which $\frac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})$ was constructed from $\hat{q}$ and $\hat{p}$, that are basic elements of the algebra, one gets better insight in how to solve the ordering problem in general. The procedure could be: $\hat{q}$ and $\hat{p}$ were multiplied as they were ordinary numbers, resulting in $\hat{q}\hat{p}$, and then some ordering procedure was applied, producing $\frac{1}{2}(\hat{q}\hat{p}+\hat{p}\hat{q})$. Taken in this way, the symmetrized product becomes a two step operation. The first step of symmetrized product is multiplication as in the case of c-numbers, with all well known properties except commutativity. The second step is the ordering procedure. We see it as the application of some superoperator $\mathbf{S}$ on the operatorial expressions consisting of $\hat{q}$'s and $\hat{p}$'s. Then, definition of the ordering procedure becomes a definition of superoperator $\mathbf{S}$ that acts on sequences of $\hat{q}$ and $\hat{p}$ assigning them, in some well defined manner, another sequences. By adopting this way of looking on symmetrized product, as we are going to do in what follows, it may be said that propositions of ordering rules given in literature differ since they (implicitly) use different superoperators - symmetrizers.

We shall use $\mathbf{S}$ defined as the linear superoperator that acts on sequences of $\hat{q}$ and $\hat{p}$ producing the sum of all different combinations of that operators and dividing the result by the number of these combinations. For example, $\mathbf{S}(\hat{q}^2\hat{p}^2) = \frac{1}{6}(\hat{q}^2\hat{p}^2+\hat{q}\hat{p}\hat{q}\hat{p}+\hat{q}\hat{p}^2\hat{q}+\hat{p}\hat{q}^2\hat{p}+\hat{p}\hat{q}\hat{p}\hat{q}+\hat{p}^2\hat{q}^2)$. In general, when $\mathbf{S}$ acts on sequence $\hat{q}\hat{p}\hat{p}\cdots\hat{p}\hat{q}\hat{q}$, where the operator of coordinate appears n times and the operator of momentum appears m times, no matter how they are ordered in this sequence, the result will be:

$$\frac{n!m!}{(n+m)!}(\hat{q}^n\hat{p}^m+\cdots+\hat{p}^m\hat{q}^n). \tag{4}$$

It is understood that in the parenthesis there should be all different combinations of $n+m$ operators. Since there are n operators of the one kind and m of the other, there should be $\frac{(n+m)!}{n!m!}$ terms in the parenthesis.

More formally, the action of $\mathbf{S}$ on operatorial sequences is defined by:

$$\mathbf{S}(\hat{q}^{a_1}\hat{p}^{b_1}\cdots\hat{q}^{a_n}\hat{p}^{b_n}) = \frac{(\sum_i a_i)!(\sum_i b_i)!}{(\sum_i a_i+\sum_i b_i)!} \sideset{}{'}\sum_{\substack{c_1,\cdots,c_m \\ d_1,\cdots,d_m \\ \sum_j c_j=\sum_i a_i \\ \sum_j d_j=\sum_i b_i}} \hat{q}^{c_1}\hat{p}^{d_1}\cdots\hat{q}^{c_m}\hat{p}^{d_m}, \tag{5}$$

where $a_i, b_i, c_j, d_j \in \mathbf{N}_o,\ i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\},\ n \in \mathbf{N},\ m = \sum_i a_i + \sum_i b_i + 1$ and where the prime over the sum indicates the absence of repeated combinations, *i.e.*, if:

$$\hat{q}^{c'_1}\hat{p}^{d'_1} \cdots \hat{q}^{c'_m}\hat{p}^{d'_m} = \hat{q}^{c''_1}\hat{p}^{d''_1} \cdots \hat{q}^{c''_m}\hat{p}^{d''_m},$$

then $c'_j = c''_j$ and $d'_j = d''_j$ for all j. (The immediate consequence of (5) is that:

$$\mathbf{S}(\hat{q}^{a_1}\hat{p}^{b_1} \cdots \hat{q}^{a_n}\hat{p}^{b_n}) = \mathbf{S}(\hat{q}^{c_1}\hat{p}^{d_1} \cdots \hat{q}^{c_m}\hat{p}^{d_m}), \tag{6}$$

whenever $\sum_i a_i = \sum_j c_j$ and $\sum_k b_k = \sum_l d_l$.) The linearity of $\mathbf{S}$ reads:

$$\mathbf{S}(k\hat{q}^{a_1}\hat{p}^{b_1} \cdots \hat{q}^{a_n}\hat{p}^{b_n} + l\hat{q}^{c_1}\hat{p}^{d_1} \cdots \hat{q}^{c_m}\hat{p}^{d_m}) =$$

$$= k\mathbf{S}(\hat{q}^{a_1}\hat{p}^{b_1} \cdots \hat{q}^{a_n}\hat{p}^{b_n}) + l\mathbf{S}(\hat{q}^{c_1}\hat{p}^{d_1} \cdots \hat{q}^{c_m}\hat{p}^{d_m}), \tag{7}$$

where $a_i, b_i, c_j, d_j \in \mathbf{N}_o,\ i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\},\ n, m \in \mathbf{N},\ k, l \in \mathbf{C}$. The third defining property of $\mathbf{S}$ is:

$$\mathbf{S}(k\hbar^m\hat{q}^{a_1}\hat{p}^{b_1} \cdots \hat{q}^{a_n}\hat{p}^{b_n}) = 0, \tag{8}$$

where $a_i, b_i \in \mathbf{N}_o,\ i \in \{1, \cdots, n\},\ n \in \mathbf{N}, k \in \mathbf{C},\ m \in \mathbf{N}$. Necessity of this property will be discussed in the next section.

For reasons that will be given in Sec. 4, the symmetrized product of observables and the partial derivatives of state $\hat{\rho}$ is defined in way which is a slight modification of the above. Precisely, (5) becomes:

$$\mathbf{S}(\hat{q}^{a_1}\hat{p}^{b_1}(\frac{\partial\hat{\rho}}{\partial\hat{r}})^{e_1} \cdots \hat{q}^{a_n}\hat{p}^{b_n}(\frac{\partial\hat{\rho}}{\partial\hat{r}})^{e_n}) =$$

$$= \frac{(\sum_i a_i)!(\sum_i b_i)!}{(\sum_i a_i + \sum_i b_i + \sum_i e_i)!} \sideset{}{'}\sum_{\substack{c_1,\cdots,c_m \\ d_1,\cdots,d_m \\ f_1,\cdots,f_m \\ \sum_j c_j = \sum_i a_i \\ \sum_j d_j = \sum_i b_i \\ \sum_j f_j = \sum_i e_i}} \hat{q}^{c_1}\hat{p}^{d_1}(\frac{\partial\hat{\rho}}{\partial\hat{r}})^{f_1} \cdots \hat{q}^{c_m}\hat{p}^{d_m}(\frac{\partial\hat{\rho}}{\partial\hat{r}})^{f_m}, \tag{9}$$

where $a_i, b_i, c_j, d_j \in \mathbf{N}_o,\ i \in \{1, \cdots, n\}, j \in \{1, \cdots, m\},\ n \in \mathbf{N},\ m = \sum_i a_i + \sum_j b_j + 1,\ e_i, e_j \in \{0, 1\}$ and $\hat{r} \in \{\hat{q}, \hat{p}\}$. Modifications of the other properties follow straightforwardly. On the other hand, it is not needed to modify above expressions for the identity operator because it can be seen as $\hat{q}^o$ or $\hat{p}^o$.

The expression (4) should be taken, we propose, as corresponding to the monomial $q^n p^m$ of classical mechanics and, since $\hat{q}^n \circ \hat{p}^m$ should represent the symmetrized product of n operators of coordinate and m operators of momentum, it was constructed from these $n+m$ operators in two steps (the first was to treat these operators as ordinary numbers, the meaning of which is to take $\hat{q}^n \hat{p}^m$, or any other sequence of these $n+m$ operators, and the second step was to act with $\mathbf{S}$ on that). Then, by applying this procedure to a bit complicated case, one can easily answer the question what shall be the symmetrized product of two monomials, let say, $\hat{q}^a \circ \hat{p}^b$ and $\hat{q}^c \circ \hat{p}^d$ (both monomials are symmetrized in the above way). Namely, the result should be the expression gained after the expressions similar to (4) were multiplied as it would be done in the c-number case and after $\mathbf{S}$ was applied on that. After the first step, one would get:

$$\frac{a!b!}{(a+b)!}\frac{c!d!}{(c+d)!}(\hat{q}^a \hat{p}^b \hat{q}^c \hat{p}^d + \cdots + \hat{p}^b \hat{q}^a \hat{p}^d \hat{q}^c),$$

where, in the parenthesis, there should be $\frac{(a+b)!}{a!b!}\frac{(c+d)!}{c!d!}$ terms on which $\mathbf{S}$ acts in the same way. The final result would be:

$$\frac{(a+c)!(b+d)!}{(a+b+c+d)!}(\hat{q}^{a+c} \hat{p}^{b+d} + \cdots + \hat{p}^{b+d} \hat{q}^{a+c}).$$

Again, in the parenthesis, there should stand only different combinations of $a+b+c+d$ operators, where $a+c$ are of the one kind and $b+d$ are of the other. In more compact notation, the above reads:

$$(\hat{q}^a \circ \hat{p}^b) \circ (\hat{q}^c \circ \hat{p}^d) = \mathbf{S}(\mathbf{S}(\hat{q}^a \cdot \hat{p}^b) \cdot \mathbf{S}(\hat{q}^c \cdot \hat{p}^d)) = \hat{q}^{a+c} \circ \hat{p}^{b+d}.$$

Consequently, the symmetrized product of quantum mechanical observables $\sum_i c_i \hat{q}^{n_i} \circ \hat{p}^{m_i}$ and $\sum_j d_j \hat{q}^{r_j} \circ \hat{p}^{s_j}$ should be:

$$(\sum_i c_i \hat{q}^{n_i} \circ \hat{p}^{m_i}) \circ (\sum_j d_j \hat{q}^{r_j} \circ \hat{p}^{s_j}) = \sum_i \sum_j c_i d_j \hat{q}^{n_i + r_j} \circ \hat{p}^{m_i + s_j}. \qquad (10)$$

The involved monomials are defined above.

3 Some properties of the symmetrized product

Our proposal of the symmetrized product is of the same complexity at the level of correspondence principle as are those given in literature: ordinary product, multiplication by numbers and summation all have to be used in order to define it.

Our proposal differs in that we find it necessary to look on the symmetrized product as on a two step operation. If it is so when the correspondence principle is under consideration, then we find it necessary to apply both of the mentioned steps in all other situations where the algebraic structure of quantum mechanics is addressed.

If one multiplies in ordinary manner operatorial expressions that were symmetrized in either way, then it does not come as surprise that result is no longer adequately ordered. Moreover, the omission at this place of the second step - the application of **S**, makes such approaches inconsistent. That is because basic operatorial polynomials, which by definition correspond to classical mechanical variables, actually are symmetrical products of several $\hat{q}$'s and $\hat{p}$'s introduced at the beginning of quantization.

However, the ordered product is not always applied in the standard quantum mechanics, see Messiah (1961). For example, the calculation of dispersion asks to take ordinary product of operators instead of any of the proposed algebraical ones. This implicates that one has to be more specific in writing, let say, $f(\hat{q},\hat{p})^2$ since it is not obvious is it $f(\hat{q},\hat{p}) \cdot f(\hat{q},\hat{p})$ or $f(\hat{q},\hat{p}) \circ f(\hat{q},\hat{p})$. There are situations where one needs the successive application of operators on states being uninterested in the algebraic aspect of quantum mechanics and *vice versa*. Needless to say, the last two expressions differ in general.

The algebra of quantum mechanical observables will be formally closed and there will be no purely algebraic obstructions with $\circ$ because the symmetrized product is a two step operation in all instances (when the correspondence with classical variables is established and when those quantum observables are multiplied) and because **S** is defined for sequences of $\hat{q}$'s and $\hat{p}$'s on the first place and not for operators in general like it is the case in Temple (!935). This means that, for the calculation of the product of general observables, it is necessary to express them like it was done on the LHS of (10). These, we believe, guarantee that one will not go out of the proposed way of symmetrization. Since all combinations of involved $\hat{q}$ and $\hat{p}$ appear on the RHS of (10), the symmetrized product of two symmetrized polynomials, which are the Hermitian, obviously will be invariant under the Hermitian conjugation.

It is necessary to declare the action of $\mathbf{S}$ on $\hbar$ for it is dimensional constant, not a complex number. Metaphorically speaking, in contrast to the other constants it "incorporates" coordinate and momentum whose numbers of appearances in expressions are important from the point of view of $\mathbf{S}$. The property (8) prevents one to end with some ambiguity in case when the commutation relation $[\hat{q}, \hat{p}] = i\hbar\hat{I}$ is used for the reexpression of operatorial sequences before the application of $\mathbf{S}$. For example, without (8), the symmetrized product of $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q})$ with itself will not be equal to the symmetrized product of $\hat{q}\hat{p} - \frac{i\hbar}{2}$ with itself, while $\frac{1}{2}(\hat{q}\hat{p} + \hat{p}\hat{q}) = \hat{q}\hat{p} - \frac{i\hbar}{2}$. In abstract considerations we are proceeding, the Planck constant can appear only due to the above commutation relation; we ignore artificial situations when $\hbar$ is introduced in expressions by hand. It could be said that, being the commutant of coordinate and momentum, $\hbar$ is the quantum analogue of zero - the commutant of classical q and p. And, since zero annihilates all sequences of q and p in $\cdot$ product, the Planck constant should do the same for sequences of $\hat{q}$ and $\hat{p}$ in $\circ$ product.

Another similarity between $\circ$ and $\cdot$ is that they are commutative. This characteristic of proposed symmetrized product follow directly from the fact that for the symmetrizer $\mathbf{S}$ only the numbers of $\hat{q}$'s and $\hat{p}$'s in a sequence on which it acts are relevant, in contrast to that how they are ordered in the sequence. These make us to believe that $\circ$ is defined in such a way to be the complete imitation of the product of classical mechanics.

Next remark is related to the situations where the form of operatorial expressions is fixed for some reasons. In Messiah (1961) it was said that ordered product of $f(\hat{\vec{r}})$ and $\hat{\vec{p}}$ has to be the one half of the anti-commutator of these two. Adapted to the present considerations, we have to show that:

$$\frac{1}{2}\left(\sum_{n=o}^{N} c_n \hat{q}^n \hat{p} + \hat{p} \sum_{n=o}^{N} c_n \hat{q}^n\right) = \sum_{n=o}^{N} c_n \hat{q}^n \circ \hat{p}, \tag{11}$$

since, with our proposal of symmetrized product, we do not want to contradict the well-known facts of standard quantum mechanics. In order to do that, we firstly transform the LHS of (11) into $\sum_{n=o}^{N} c_n \hat{q}^n \hat{p} - \frac{i\hbar}{2} \frac{\partial}{\partial \hat{q}} \sum_{n=o}^{N} c_n \hat{q}^n$

and then reexpress each term on the RHS of (11) in the following way:

$$\sum_{n=o}^{N} c_n \frac{1}{n+1}(\hat{p}\hat{q}^n + \hat{q}\hat{p}\hat{q}^{n-1} + \cdots + \hat{q}^{n-1}\hat{p}\hat{q} + \hat{q}^n\hat{p}) =$$

$$= \sum_{n=o}^{N} c_n \frac{1}{n+1}(\hat{q}^n\hat{p} - i\hbar n\hat{q}^{n-1} + \hat{q}(\hat{q}^{n-1}\hat{p} - i\hbar(n-1)\hat{q}^{n-2}) + \cdots + \hat{q}^{n-1}(\hat{q}\hat{p} - i\hbar) +$$

$$+\hat{q}^n\hat{p}) = \sum_{n=o}^{N} c_n\hat{q}^n\hat{p} - i\hbar\sum_{n=o}^{N} c_n \frac{1}{n+1}(n + (n-1) + \cdots + 1)\hat{q}^{n-1} =$$

$$= \sum_{n=o}^{N} c_n\hat{q}^n\hat{p} - \frac{i\hbar}{2}\sum_{n=o}^{N} c_n n\hat{q}^{n-1} =$$

$$= \sum_{n=o}^{N} c_n\hat{q}^n\hat{p} - \frac{i\hbar}{2}\frac{\partial}{\partial\hat{q}}\sum_{n=o}^{N} c_n\hat{q}^n.$$

So, our proposal of symmetrized product is not in conflict with the mentioned demand since both sides of (11) are equal. Up to our knowledge, this is the only situation where the form of observable is *a priori* fixed.

Finally, one can show that:

$$\frac{\partial}{\partial\hat{q}}(\hat{q}^n \circ \hat{p}^m) = n\hat{q}^{n-1} \circ \hat{p}^m, \tag{12}$$

the meaning of which is that the proposed algebraic structure of quantum mechanics is formally closed under the partial derivations - the type of symmetrization is saved under the action of $\frac{\partial}{\partial\hat{q}}$ and $\frac{\partial}{\partial\hat{p}}$. This will be important for redefinition of the Lie bracket of quantum mechanics that we are going to discuss in the next section.

For this purpose, one has to show that, after the partial derivation of $\hat{q}^n \circ \hat{p}^m$, there will be the sum of all $\frac{(n-1+m)!}{(n-1)!m!}$ different combinations of $n-1$ operators of coordinate and m operators of momentum and that each of these combinations will appear $n+m$ times, so the multiplicative factor $\frac{n!m!}{(n+m)!}$, standing in front of the sum and coming from $\hat{q}^n \circ \hat{p}^m$, will be regularized.

It is trivial fact that, after the partial derivation of sequences of n operators of coordinate and m operators of momentum (that appear in $\hat{q}^n \circ \hat{p}^m$), there will be sequences of $n-1$ operators of coordinate and m operators of momentum (which is asked by $\hat{q}^{n-1} \circ \hat{p}^m$). After the partial derivation of $\hat{q}^n \circ \hat{p}^m$, there will be the sum of all different combinations of $n-1$ operators of coordinate and m operators of momentum since, by definition, in $\hat{q}^n \circ \hat{p}^m$, all different combinations of appropriate operators are present. Each of these $\frac{(n+m-1)!}{(n-1)!m!}$ different combinations needed for $\hat{q}^{n-1} \circ \hat{p}^m$ will follow after the derivation of many different combinations - sequences, appearing in $\hat{q}^n \circ \hat{p}^m$. All those sequences of $\hat{q}^n \circ \hat{p}^m$, that will give after derivation a particular sequence of $\hat{q}^{n-1} \circ \hat{p}^m$, have one more $\hat{q}$. This "extra" $\hat{q}$, that is going to be annihilated by derivation, can be on $n+m$ different places when those sequences are compared to the chosen sequence of $\hat{q}^{n-1} \circ \hat{p}^m$. (In other words, the sequences of $\hat{q}^n \circ \hat{p}^m$ that will be transformed by $\frac{\partial}{\partial \hat{q}}$ in the chosen one differ from it only in one $\hat{q}$, all other $\hat{q}$'s and $\hat{p}$'s are equally ordered. To visualize this, it is helpful to consider, for instance, $\hat{q}^{n-1}\hat{p}^m$; it follows after derivation of: $\hat{q}^n\hat{p}^m$, $\hat{q}^{n-1}\hat{p}\hat{q}\hat{p}^{m-1}$, $\cdots$, $\hat{q}^{n-1}\hat{p}^{m-1}\hat{q}\hat{p}$ and $\hat{q}^{n-1}\hat{p}^m\hat{q}$ that are present in $\hat{q}^n \circ \hat{p}^m$. The mentioned extra $\hat{q}$ in $\hat{q}^n\hat{p}^m$ can be each of n operators of coordinate, like it appears on n different places. In the next $m-1$ sequences, it is the $\hat{q}$ among operators of momentum that is extra and, in the last combination, it $\hat{q}$ standing on the right of $\hat{p}^m$.) This means that each sequence of $\hat{q}$'s and $\hat{p}$'s needed for $\hat{q}^{n-1} \circ \hat{p}^m$ will follow $n+m$ times after the application of $\frac{\partial}{\partial \hat{q}}$ on $\hat{q}^n \circ \hat{p}^m$. In this way one can convince oneself that (12) holds and, by proceeding in similar manner for the partial derivation with respect to the momentum and in the case of polynomials, one can show that the algebra of observables is formally closed under the action of $\frac{\partial}{\partial \hat{q}}$ and $\frac{\partial}{\partial \hat{p}}$.

4 The redefined Lie bracket of quantum mechanics

The symmetrized product we have proposed can be used in the classical mechanics instead of the standard algebraic multiplication of variables since the ordering rules have no effect in the case of the commutative algebra. So, one can say that the algebraic product of both mechanics is one and the same. Then, the difference between them is in the Lie bracket and, in what follows, we shall try to remedy this by intervening on this bracket of quantum mechanics. We propose the substitution of the commutator divided by $i\hbar$ with the operatorial form of Poisson bracket:

$$\{f(\hat{q},\hat{p}),g(\hat{q},\hat{p})\}_{\mathbf{S}} = \frac{\partial f(\hat{q},\hat{p})}{\partial \hat{q}} \circ \frac{\partial g(\hat{q},\hat{p})}{\partial \hat{p}} - \frac{\partial g(\hat{q},\hat{p})}{\partial \hat{q}} \circ \frac{\partial f(\hat{q},\hat{p})}{\partial \hat{p}}. \tag{13}$$

Nota bene, in difference to (1), the operators are involved in the last expression and symmetrized product stays instead of the ordinary one.

Before one addresses the obstruction to quantization mentioned in Sec. 1 and before reexpresses dynamical equation of quantum mechanics, one has to convince oneself that the just defined "symmetrized" Poisson bracket has all properties of the Lie product. To do this, one can proceed as follows. That $\{\ ,\ \}_{\mathbf{S}}$ is linear one easily concludes remembering that the partial derivations and symmetrized product are linear operations. That it is anti-symmetric one can see directly from definition. By comparing the operatorial expression under consideration with the adequate one of classical mechanics, one finds that the partial derivations and symmetrized products of symmetrized operatorial expressions within $\{\ ,\ \}_{\mathbf{S}}$ in complete imitate the partial derivations and ordinary products of c-number functions in $\{\ ,\ \}$. So, the confirmation of the Jacoby identity rests on the analogy - each step of the calculation in the case of operators has the corresponding one in the c-number case.

Due to mentioned analogies, the symmetrized Poisson bracket would behave as derivative:

$$\{f(\hat{q},\hat{p}),g(\hat{q},\hat{p})\circ h(\hat{q},\hat{p})\}_{\mathbf{S}} =$$
$$= \{f(\hat{q},\hat{p}),g(\hat{q},\hat{p})\}_{\mathbf{S}} \circ h(\hat{q},\hat{p}) + g(\hat{q},\hat{p}) \circ \{f(\hat{q},\hat{p}),h(\hat{q},\hat{p})\}_{\mathbf{S}}, \tag{14}$$

This can be checked by taking three monomials in $\hat{q}$ and $\hat{p}$. For $f(\hat{q},\hat{p}) = \hat{q}^{a_1} \circ \hat{p}^{a_2}$, $g(\hat{q},\hat{p}) = \hat{q}^{b_1} \circ \hat{p}^{b_2}$ and $h(\hat{q},\hat{p}) = \hat{q}^{c_1} \circ \hat{p}^{c_2}$, both sides of (14) will be $(a_1(b_2+c_2) - a_2(b_1+c_1))\hat{q}^{a_1+b_1+c_1-1} \circ \hat{p}^{a_2+b_2+c_2-1}$.

If one takes ordinary multiplied $g(\hat{q},\hat{p})$ and $h(\hat{q},\hat{p})$, then:

$$\{f(\hat{q},\hat{p}),g(\hat{q},\hat{p})\cdot h(\hat{q},\hat{p})\}_{\mathbf{S}} \neq$$

$$\{f(\hat{q},\hat{p}),g(\hat{q},\hat{p})\}_{\mathbf{S}} \cdot h(\hat{q},\hat{p}) + g(\hat{q},\hat{p}) \cdot \{f(\hat{q},\hat{p}),h(\hat{q},\hat{p})\}_{\mathbf{S}},$$

in general. One and the same product has to be used for all multiplications in $\{\ ,\ \}_{\mathbf{S}}$ for the Leibniz rule to be satisfied.

The generalization of the symmetrized Poisson bracket for more than one degree of freedom is influenced by requirements coming from physics, see Prvanović and Marić (2000) and references therein. This topic we shall consider in the forthcoming article belonging to the series concerned with the foundation of theory of hybrid systems.

Given remarks make trivial the problem of possibility for obstruction to quantization based on the symmetrized Poisson bracket as the Lie product. If there was some equation for classical variables, then the same equation will hold for their quantum counterparts since the symmetrized product and the symmetrized Poisson bracket imitate the adequate operations in c-number case. Said in more descriptive way, the equation in quantum mechanics will hold since it differs from the corresponding equation of classical mechanics only in that there are hats above coordinate and momentum and there is $\circ$ instead of $\cdot$. Consequently, this quantization is, we believe, unambiguous, *i.e.*, obstruction free *in toto*.

Can the symmetrized Poisson bracket substitute the commutator in von Neumann equation (3) is the last question that we are going to address. Related to this are: how the operators representing states of quantum system can be expressed as depending on some functions of $\hat{q}$ and $\hat{p}$ and how should the symmetrizer $\mathbf{S}$ act on sequences formed of $\hat{q}$, $\hat{p}$ and partial derivatives of the state $\hat{\rho}$. With the help of:

$$|q'\rangle\langle q'| = \int \delta(q-q')|q\rangle\langle q|dq = \delta(\hat{q}-q'), \tag{15}$$

and:

$$|q''\rangle\langle q'| = e^{\frac{1}{i\hbar}(q''-q')\hat{p}}|q'\rangle\langle q'| = e^{\frac{1}{i\hbar}(q''-q')\hat{p}} \cdot \delta(\hat{q}-q'), \tag{16}$$

one immediately finds that the pure state in general, *i.e.*, $|\psi\rangle = \int \psi(q)|q\rangle dq$, can be expressed as:

$$|\psi\rangle\langle\psi| = \int\int \psi(q)\psi^*(q')|q\rangle\langle q'|dqdq' =$$

$$= \int\int \psi(q)\psi^*(q')e^{\frac{1}{i\hbar}(q''-q')\hat{p}} \cdot \delta(\hat{q}-q')dqdq'. \tag{17}$$

But, $\delta(\hat{q}-q')$ is neither polynomial nor analytical function of $\hat{q}$. This means that it can not be expressed in the form $\sum_n c_n \hat{q}^n$ which is necessary according to (5) for a direct calculation of the symmetrized product. Supplementary defining property of the ordering rule for a novel situation, when $\mathbf{S}$ acts on sequences of $\hat{q}$, $\hat{p}$ and the partial derivatives of $\delta(\hat{q}-q')$ and/or the general state, are certainly needed. This property is (9) the meaning of which is that the partial derivatives of states are entities different from $\hat{q}$ and $\hat{p}$.

If the symmetrical product of $\hat{q}^n \circ \hat{p}^m$ and partial derivative of $|\psi\rangle\langle\psi|$ is to be calculated, one should look for $\frac{(n+m+1)!}{n!m!}$ distinct combinations of involved operators. Concretely:

$$\hat{p} \circ \frac{\partial|\psi\rangle\langle\psi|}{\partial\hat{q}} = \frac{1}{2}(\hat{p}\frac{\partial|\psi\rangle\langle\psi|}{\partial\hat{q}} + \frac{\partial|\psi\rangle\langle\psi|}{\partial\hat{q}}\hat{p}), \tag{18}$$

and:

$$\hat{q}^n \circ \frac{\partial|\psi\rangle\langle\psi|}{\partial\hat{p}} =$$

$$\frac{1}{n+1}(\hat{q}^n\frac{\partial|\psi\rangle\langle\psi|}{\partial\hat{p}} + \hat{q}^{n-1}\frac{\partial|\psi\rangle\langle\psi|}{\partial\hat{p}}\hat{q} + \cdots + \hat{q}\frac{\partial|\psi\rangle\langle\psi|}{\partial\hat{p}}\hat{q}^{n-1} + \frac{\partial|\psi\rangle\langle\psi|}{\partial\hat{p}}\hat{q}^n). \tag{19}$$

Then, one can easily show that:

$$\{\frac{\hat{p}^2}{2m} + V(\hat{q}), |\psi\rangle\langle\psi|\}_{\mathbf{S}} = \frac{1}{i\hbar}[\frac{\hat{p}^2}{2m} + V(\hat{q}), |\psi\rangle\langle\psi|], \tag{20}$$

where $V(\hat{q}) = \sum_n c_n\hat{q}^n$. Instead of proving (20) in the coordinate representation, one can simply substitute $\frac{\partial}{\partial\hat{q}}$ and $\frac{\partial}{\partial\hat{p}}$ that appear on the LHS of (20) with $\frac{1}{i\hbar}[\ ,\hat{p}]$ and $\frac{1}{i\hbar}[\hat{q},\ ]$, respectively, and, by using (18) and (19), after few elementary steps one will find that the LHS and RHS of (20) are equal. Despite of being less interesting for physics, let us pay more attention on:

$$\{F(\hat{q},\hat{p}), |\psi\rangle\langle\psi|\}_{\mathbf{S}} = \frac{1}{i\hbar}[F(\hat{q},\hat{p}), |\psi\rangle\langle\psi|], \tag{21}$$

where $F(\hat{q},\hat{p})$ is the general element of the quantum mechanical algebra. The analysis of (21) can start with considerations of some monomial $\hat{q}^n \circ \hat{p}^m$. In order to obtain the LHS of (21), according to (12), one, firstly, has to multiply symmetrically $n\hat{q}^{n-1} \circ \hat{p}^m$ and $\frac{\partial\hat{\rho}}{\partial\hat{p}}$, where $\hat{\rho} = |\psi\rangle\langle\psi|$, then to multiply symmetrically $m\hat{q}^n \circ \hat{p}^{m-1}$ and $-\frac{\partial\hat{\rho}}{\partial\hat{q}}$ and, finally, to add these two. Due to (9), the LHS of (21) then becomes:

$$\frac{n!m!}{(n+m)!}(\hat{q}^{n-1}\hat{p}^m\frac{\partial\hat{\rho}}{\partial\hat{p}} + \cdots + \frac{\partial\hat{\rho}}{\partial\hat{p}}\hat{p}^m\hat{q}^{n-1} - (\hat{q}^n\hat{p}^{m-1}\frac{\partial\hat{\rho}}{\partial\hat{q}} + \cdots + \frac{\partial\hat{\rho}}{\partial\hat{q}}\hat{p}^{m-1}\hat{q}^n)). \tag{22}$$

After substituting $\frac{\partial\hat{\rho}}{\partial\hat{p}}$ with $\frac{1}{i\hbar}[\hat{q},\hat{\rho}]$ and $\frac{-\partial\hat{\rho}}{\partial\hat{q}}$ with $\frac{1}{i\hbar}[\hat{p},\hat{\rho}]$, one should look how to simplify (22). Some terms in (22) will be of the form $\hat{A}\hat{q}(\hat{q}\hat{\rho})\hat{q}\hat{B}$, where

$\hat{A}$ and $\hat{B}$ represent (different) sequences of $\hat{q}$'s and $\hat{p}$'s and $(\hat{q}\hat{\rho})$ means that these two come from the commutator $[\hat{q}, \hat{\rho}]$. But, such terms will be canceled by $-\hat{A}\hat{q}\hat{q}(\hat{\rho}\hat{q})\hat{B}$ (which certainly should appear in (22)), where the minus sign comes from the commutator. Proceeding in this way for all other forms, one can find that many terms in (22) will mutually cancel each other. This holds for all terms except those where $\hat{\rho}$ stands at the beginning or at the end of the sequence. Consequently, (22) is equal to:

$$\frac{1}{i\hbar}\frac{n!m!}{(n+m)!}((\hat{q}^n\hat{p}^m + \cdots + \hat{p}^m\hat{q}^n)\hat{\rho} - \hat{\rho}(\hat{q}^n\hat{p}^m + \cdots + \hat{p}^m\hat{q}^n)),$$

which is nothing else than the RHS of (21) for the considered monomial. Then, due to the linearity of commutator and symmetrized Poisson bracket, (21) will hold for general $F(\hat{q}, \hat{p})$, too.

It should be noticed that (20) will not hold for $V(\hat{q}) = \hat{q}^n$ when $n \geq 3$ if the symmetrized product within Poisson bracket is taken to be the one half of the anti-commutator of involved operators. This is why we have defined $\mathbf{S}$ to be the superoperator that produces (19), *i.e.*, defined by (9).

From (20) and (21), it follows that the dynamical equation of quantum mechanics can be reexpressed. Obviously, resulting equation is operatorial version of the Liouville equation:

$$\frac{\partial\rho(\hat{q}, \hat{p}, t)}{\partial t} = \{H(\hat{q}, \hat{p}), \rho(\hat{q}, \hat{p}, t)\}_{\mathbf{S}}. \tag{23}$$

It is understood that the Hamiltonian is $H(\hat{q}, \hat{p}) = \sum_i c_i \hat{q}^{n_i} \circ \hat{p}^{m_i}$.

5 Concluding remarks

For the introduction of the symmetrized product that we have proposed it was necessary to look on the operators of coordinate and momentum as on the basic elements. The special position that coordinate and momentum have in mechanics has manifested itself in that all physically meaningful entities have to be defined, or expressed, via them. The algebraic and Lie algebraic products have to be defined, on the first place, with respect to the coordinate and momentum and not for the operators in general the particular examples of which would be $\hat{q}$ and $\hat{p}$. Only after observables are expressed in form of the functions of $\hat{q}$ and $\hat{p}$, then their algebraic or Lie algebraic product can be found. The numbers of appearances of $\hat{q}$'s and $\hat{p}$'s in operatorial sequences play the crucial role in such situations, just like degrees of q and p do in the c-number case.

As we have stressed, the only way to make the approach consistent and, we believe, to avoid ambiguities is to apply the same ordering procedure in all situations. If it was seen as a two step operation at the level of the correspondence principle, then it should be treated in the same manner in other occasions as well. And, moreover, we believe that the ordering procedure has to be taken as a two step operation if the Hermiticity of observables is requested.

The unavoidable second step of all multiplications, *i.e.*, the symmetrizer **S**, due to properties that it produces all distinct combinations of coordinate and momentum and annihilates $\hbar$, makes the approach obstruction free. When the operators introduced at the beginning of quantization are looked from the point of view of either algebra or Lie algebra, there will be no contradictory statements because **S** within $\circ$ and $\{\ ,\ \}_{\mathbf{S}}$ forms these two operations to be the complete imitation of the corresponding ones of classical mechanics.

At last, let us summarize in brief our proposition for the quantization. The basic elements of classical mechanical algebra: q, p and 1, should be mapped into: $\hat{q}$, $\hat{p}$ and $\hat{I}$, respectively, where $[\hat{q},\hat{p}] = \frac{1}{i\hbar}\hat{I}$. The general element $\sum_i c_i q^{n_i} \cdot p^{m_i}$ should be mapped in $\sum_i c_i \hat{q}^{n_i} \circ \hat{p}^{m_i}$. The ordinary product of $f(q,p) = \sum_i c_i q^{n_i} \cdot p^{m_i}$ and $g(q,p) = \sum_j d_j q^{r_j} \cdot p^{s_j}$ should be translated in (10), and their Lie product (1) should become (13). The dynamical equations for states, which are expressible via coordinate and momentum in both mechanics, should be (2) in classical mechanics and (23) in quantum mechanics.

If we neglect for the moment that classical mechanics is represented in this article with c-number functions, while quantum mechanics uses operatorial functions, then it could be said not only that classical and quantum mechanics are equivalent regarding the mathematical structures, but the algebraic and Lie algebraic products have the same realization for both mechanics since the symmetrized product and symmetrized Poisson bracket can be used without any problem instead of the ordinary ones in classical mechanics, as well. This we are going to use in the next article devoted to the formulation of theory of hybrid systems.

References

[1] Arens, R., and Babit, D., (1965). *Journal of Mathematical Physics*, **6**, 1071.

[2] Chernoff, P. R., (1995). *Journal of Functional Analysis*, **130**, 255.

[3] Cohen, L., (1966). *Journal of Mathematical Physics*, **7**, 781.

[4] Dirac, P. A. M., (1958). *Principles of quantum mechanic*, Oxford University Press, New York.

[5] Emch, G. G., (1972). *Algebraic methods in statistical mechanics and quantum field theory*, Wiley, New York.

[6] Gotay, M. J., (1980). *International Journal of Theoretical Physics*, **19**, 139.

[7] Gotay, M. J., Grundling, H. B., and Tuynman, G. M., (1996). *Journal of Nonlinear Sciences*, **6**, 469.

[8] Joseph, A., (1970). *Communications in Mathematical Physics*, **17**, 210.

[9] Kerner, E. H., and Sutcliffe, W. G., (1970). *Journal of Mathematical Physics*, **11**, 391.

[10] Messiah, A., (1961). *Quantum mechanics I*, North-Holland, Amsterdam.

[11] Prvanović, S., and Marić, Z., (2000). *International Journal of Theoretical Physics*, **39**, 1469.

[12] Shewell, J. R., (1959). *American Journal of Physics*, **27**, 16.

[13] Temple, G., (1935). *Nature*, **135**, 957.